

DEGREE DISTRIBUTIONS FOR A CLASS OF CIRCULANT GRAPHS

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ABSTRACT. We characterize the equivalence and the weak equivalence of Cayley graphs for a finite group \mathcal{A} . Using these characterizations, we find degree distribution polynomials for weak equivalence of some graphs including 1) circulant graphs of prime power order, 2) circulant graphs of order $4p$, 3) circulant graphs of square free order and 4) Cayley graphs of order p or $2p$. As an application, we find an enumeration formula for the number of weak equivalence classes of circulant graphs of prime power order, order $4p$ and square free order and Cayley graphs of order p or $2p$.

1. INTRODUCTION

Let \mathcal{A} be a finite group with identity e and let Ω be a generating set for \mathcal{A} with properties that $\Omega = \Omega^{-1}$ and $e \notin \Omega$, where $\Omega^{-1} = \{x^{-1} \mid x \in \Omega\}$. The *Cayley graph* $C(\mathcal{A}, \Omega)$ is a simple graph whose vertex set and edge set are defined as follows:

$$V(C(\mathcal{A}, \Omega)) = \mathcal{A} \text{ and } E(C(\mathcal{A}, \Omega)) = \{\{g, h\} \mid g^{-1}h \in \Omega\}.$$

Because of their rich connections with a broad range of areas, Cayley graphs have been in the center of the research in graph theory [3, 7, 24, 25]. Spectral estimations of Cayley graphs have been studied [4, 11]. It plays a key role in the study of the geometry of hyperbolic groups [15]. Recently, Li has found wonderful results on edge-transitive Cayley graphs [19, 20]. For standard terms and notations, we refer to [12].

The Cayley graph $C(\mathcal{A}, \Omega)$ admits a natural \mathcal{A} -action, $\cdot : \mathcal{A} \times C(\mathcal{A}, \Omega) \rightarrow C(\mathcal{A}, \Omega)$ defined by $g \cdot g' = gg'$ for all $g, g' \in \mathcal{A}$. A graph Γ with an \mathcal{A} -action is called an \mathcal{A} -graph. So, every Cayley graph $C(\mathcal{A}, \Omega)$ is an \mathcal{A} -graph. A graph isomorphism $f : \Gamma_1 \rightarrow \Gamma_2$ between two \mathcal{A} -graphs Γ_1 and Γ_2 is a *weak equivalence* if there exists a group automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ such that $f(g \cdot u) = \alpha(g) \cdot f(u)$ for all $g \in \mathcal{A}$ and $u \in V(\Gamma_1)$. When α is the identity automorphism, we say that f is an *equivalence*. If there is a weak equivalence between \mathcal{A} -graphs Γ_1 and Γ_2 , we say Γ_1 and Γ_2 are *weak equivalent*. Similarly, if there is an equivalence between \mathcal{A} -graphs Γ_1 and Γ_2 , we say Γ_1 and Γ_2 are *equivalent*. Enumerations of the equivalence classes and weak equivalence classes of some graphs have been studied [9, 16].

In particular, the isomorphism problem of Cayley graphs has been studied by several authors [5, 10, 17]. However, one classical isomorphism problem on Cayley graphs is a conjecture rose by Ádám [1] that two Cayley graphs of \mathbb{Z}_n are isomorphic if and only if they are isomorphic by a group automorphism of \mathbb{Z}_n . This conjecture was first disproven by Elpas and Turner [8]. After that, a particular attention has been paid to determine which group \mathcal{A} has the property that two Cayley graphs of \mathcal{A} are isomorphic if and only

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if they are isomorphic by a group automorphism of \mathcal{A} . Such a group is called a *CI-group*. For cyclic group \mathbb{Z}_n , CI-groups were completely classified by Muzychuk [21, 22] that a cyclic group of order n is a CI-group if and only if $n = 8, 9, 18, m, 2m$ or $4m$ where m is odd and square-free. Therefore, for any square-free number n , the number of non-isomorphic connected k -regular Cayley graphs of \mathbb{Z}_n is equal to the coefficient $a_k(\mathbb{Z}_n)$ of x^k in $\Psi_{\mathbb{Z}_n}^w(x)$ which is the degree distribution polynomial of weak equivalence classes of Cayley graphs whose underlying group is \mathbb{Z}_n and the number of non-isomorphic connected Cayley graphs of \mathbb{Z}_n is equal to $\Psi_{\mathbb{Z}_n}^w(1)$. Li has a wonderful survey for the isomorphism problem of Cayley graphs [18].

A *circulant* graph is a graph whose automorphism group of the graph includes a cyclic subgroup which acts transitively on vertex set of the graph. The isomorphism problem of circulant graphs had been studied by several authors [2, 6] and completely solved by Muzychuk [23].

In present article, we deal with the weak equivalence classes of circulant graphs. We first find a characterization of the equivalence and the weak equivalence of Cayley graphs for a finite group \mathcal{A} . As the main result of the article, we find the degree distribution polynomials for the weak equivalence classes of circulant graphs of prime power order or square free order. As an application, we find an enumeration formula for the number of the weak equivalence classes of circulant graphs of prime power order, order $4p$ and square free order and the number of the weak equivalence classes of Cayley graphs of order p or $2p$.

The outline of this paper is as follows. In Section 2, we characterize the equivalence and the weak equivalence of Cayley graphs for a finite group \mathcal{A} . In Section 3, we find some computation formulae for degree distribution polynomials. Combining results in these two sections, we find the degree distribution polynomials for the weak equivalence classes of circulant graphs of prime power order and circulant graphs of square free order in Section 4. At last we find the degree distribution polynomials for the weak equivalence classes of Cayley graphs of order p or $2p$ in Section 5.

2. A CHARACTERIZATION OF CAYLEY GRAPHS

Our definition of a weak equivalence between two Cayley graphs can be interpreted as a color-consistence and a direction preserving graph isomorphism [12, Section 1.2.4].

Theorem 2.1. *Let $C(\mathcal{A}, \Omega)$ and $C(\mathcal{A}, \Omega')$ be two Cayley graphs. The followings are equivalent.*

- (1) $C(\mathcal{A}, \Omega)$ and $C(\mathcal{A}, \Omega')$ are weakly equivalent,
- (2) There exists an automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ such that $\alpha(\Omega) = \Omega'$.

Proof. (1) \Rightarrow (2): Let $f : C(\mathcal{A}, \Omega) \rightarrow C(\mathcal{A}, \Omega')$ be a weak equivalence. Then there exists a group automorphism $\tau : \mathcal{A} \rightarrow \mathcal{A}$ such that $f(g) = \tau(g)f(e)$ for each $g \in \mathcal{A}$. Let $x \in \Omega$. Then $\{e, x\}$ is an edge in $C(\mathcal{A}, \Omega)$. Since $\{f(e), f(x)\}$ is an edge in $C(\mathcal{A}, \Omega')$, $f(e)^{-1}f(x) = f(e)^{-1}\tau(x)f(e)$ is an element of Ω' . Hence the map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\alpha(g) = f(e)^{-1}\tau(g)f(e)$ is a group automorphism such that $\alpha(\Omega) = \Omega'$.

(2) \Rightarrow (1): Let $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ be a group automorphism such that $\alpha(\Omega) = \Omega'$. We define $f : C(\mathcal{A}, \Omega) \rightarrow C(\mathcal{A}, \Omega')$ by $f(g) = \alpha(g)$. If $\{g, h\}$ is an edge in $C(\mathcal{A}, \Omega)$, then $g^{-1}h \in \Omega$

and $f(g)^{-1}f(h) = \alpha(g)^{-1}\alpha(h) = \alpha(g^{-1}h) \in \alpha(\Omega) = \Omega'$. Hence f is a graph isomorphism such that $f(gg') = \alpha(gg') = \alpha(g)\alpha(g') = \alpha(g)f(g')$, i.e., f is a weak equivalence. \square

By using a similar method in the proof of Theorem 2.1, we can have the following theorem.

Theorem 2.2. *Let $C(\mathcal{A}, \Omega)$ and $C(\mathcal{A}, \Omega')$ be two Cayley graphs. The followings are equivalent.*

- (1) $C(\mathcal{A}, \Omega)$ and $C(\mathcal{A}, \Omega')$ are equivalent,
- (2) Ω and Ω' are conjugate in \mathcal{A} , i.e., there exists an element $\gamma \in \mathcal{A}$ such that $\gamma^{-1}\Omega\gamma = \Omega'$.

Proof. (1) \Rightarrow (2): Let $f : C(\mathcal{A}, \Omega) \rightarrow C(\mathcal{A}, \Omega')$ be an equivalence and let $x \in \Omega$. Then $\{e, x\}$ is an edge in $C(\mathcal{A}, \Omega)$. Since $\{f(e), f(x)\}$ is an edge in $C(\mathcal{A}, \Omega')$, $f(e)^{-1}f(x) = f(e)^{-1}xf(e)$ is an element of Ω' . Hence $f(e)^{-1}\Omega f(e) = \Omega'$.

(2) \Rightarrow (1): Let γ be an element of \mathcal{A} such that $\gamma^{-1}\Omega\gamma = \Omega'$. We define $f : C(\mathcal{A}, \Omega) \rightarrow C(\mathcal{A}, \Omega')$ by $f(g) = g\gamma$. If $\{g, h\}$ is an edge in $C(\mathcal{A}, \Omega)$, then $g^{-1}h \in \Omega$ and $f(g)^{-1}f(h) = (g\gamma)^{-1}(h\gamma) = (\gamma^{-1}g^{-1})(h\gamma) = \gamma^{-1}(g^{-1}h)\gamma \in \gamma^{-1}\Omega\gamma = \Omega'$. Hence f is a graph isomorphism such that $f(gg') = (gg')\gamma = g(g'\gamma) = gf(g')$, i.e., f is an equivalence. \square

For a finite group \mathcal{A} , let $G(\mathcal{A}) = \{\Omega \subset \mathcal{A} : \Omega^{-1} = \Omega, < \Omega > = \mathcal{A}, e \notin \Omega\}$. Notice that $G(\mathcal{A})$ contains all equivalence classes of Cayley graphs $C(\mathcal{A}, \Omega)$. Furthermore, any subgroup of group automorphisms of \mathcal{A} admits a natural action on $G(\mathcal{A})$ by $\alpha \cdot \Omega = \alpha(\Omega)$. By Theorem 2.1, $\mathcal{E}^w(\mathcal{A})$, the number of weak equivalence classes of Cayley graphs $C(\mathcal{A}, \Omega)$, is equal to the number of orbits of the $\text{Aut}(\mathcal{A})$ action on $G(\mathcal{A})$, where $\text{Aut}(\mathcal{A})$ is the group of all group isomorphisms of \mathcal{A} . Similarly, one can see that the number $\mathcal{E}(\mathcal{A})$ of the equivalence classes of Cayley graphs $C(\mathcal{A}, \Omega)$ is equal to the number of orbits of the $\text{Inn}(\mathcal{A})$ action on $G(\mathcal{A})$ by Theorem 2.2, where $\text{Inn}(\mathcal{A})$ is the group of all inner automorphisms of \mathcal{A} .

For a finite group \mathcal{A} , let $a_k^w(\mathcal{A})$ (resp., $a_k(\mathcal{A})$) be the number of the weak equivalence (resp., equivalence) classes of Cayley graphs $C(\mathcal{A}, \Omega)$ with degree k . We call the polynomial

$$\Psi_{\mathcal{A}}^w(x) = \sum_{k=1}^{|\mathcal{A}|-1} a_k^w(\mathcal{A})x^k \quad (\text{resp., } \Psi_{\mathcal{A}}(x) = \sum_{k=1}^{|\mathcal{A}|-1} a_k(\mathcal{A})x^k)$$

the *degree distribution polynomial of the weak equivalence classes* (equivalence classes, respectively) of Cayley graphs whose underlying group is \mathcal{A} . Notice that $\Psi_{\mathcal{A}}^w(1) = \mathcal{E}^w(\mathcal{A})$ and $\Psi_{\mathcal{A}}(1) = \mathcal{E}(\mathcal{A})$. For convenience, for any finite group \mathcal{A} and any automorphism $\alpha \in \text{Aut}(\mathcal{A})$, let $\text{Fix}_{\alpha}(\mathcal{A}) = \{\Omega \in G(\mathcal{A}) : \alpha(\Omega) = \Omega\}$. Now the following theorem comes from the Burnside lemma.

Theorem 2.3. *Let \mathcal{A} be a finite group. Then we have*

$$\Psi_{\mathcal{A}}^w(x) = \frac{1}{|\text{Aut}(\mathcal{A})|} \sum_{\alpha \in \text{Aut}(\mathcal{A})} \left(\sum_{\Omega \in \text{Fix}_{\alpha}(\mathcal{A})} x^{|\Omega|} \right),$$

and

$$\Psi_{\mathcal{A}}(x) = \frac{1}{|\text{Inn}(\mathcal{A})|} \sum_{\alpha \in \text{Inn}(\mathcal{A})} \left(\sum_{\Omega \in \text{Fix}_{\alpha}(\mathcal{A})} x^{|\Omega|} \right).$$

In order to compute $\sum_{\Omega \in \text{Fix}_\alpha(\mathcal{A})} x^{|\Omega|}$, we will find a formula in terms of the Möbius function defined on the subgroup lattice of \mathcal{A} . The Möbius function assigns an integer $\mu(K)$ to each subgroup K of \mathcal{A} by the recursive formula

$$\sum_{H \geq K} \mu(H) = \delta_{K, \mathcal{A}} = \begin{cases} 1 & \text{if } K = \mathcal{A}, \\ 0 & \text{if } K < \mathcal{A}. \end{cases}$$

Jones [13, 14] used such functions to count the normal subgroups of a surface group and a crystallographic group, and applied it to count certain covering surfaces.

For convenience, let $S(\mathcal{A}) = \{\Omega \subset \mathcal{A} : \Omega = \Omega^{-1}, e \notin \Omega\}$ for a finite group \mathcal{A} and for any subgroup K of \mathcal{A} , let $S(K) = \{\Omega \subset K : \Omega = \Omega^{-1}, e \notin \Omega\}$. Then we can see that $S(\mathcal{A}) = \bigcup_{K \leq \mathcal{A}} G(K)$, and that

$$\sum_{\Omega \in S(\mathcal{A}), \alpha(\Omega) = \Omega} x^{|\Omega|} = \sum_{K \leq \mathcal{A}} \left(\sum_{\Omega \in G(K), \alpha(\Omega) = \Omega} x^{|\Omega|} \right) = \sum_{K \leq \mathcal{A}} \left(\sum_{\Omega \in \text{Fix}_\alpha(K)} x^{|\Omega|} \right).$$

Now, the following lemma easily can be obtained by the Möbius inversion.

Lemma 2.4. *Let \mathcal{A} be a finite group and let $\alpha \in \text{Aut}(\mathcal{A})$. Then*

$$\sum_{\Omega \in \text{Fix}_\alpha(\mathcal{A})} x^{|\Omega|} = \sum_{K \leq \mathcal{A}} \mu(K) \left(\sum_{\Omega \in S(K), \alpha(\Omega) = \Omega} x^{|\Omega|} \right).$$

Using Lemma 2.4, we can rephrase Theorem 2.3 as follows.

Theorem 2.5. *Let \mathcal{A} be finite group. Then we have*

$$\Psi_{\mathcal{A}}^w(x) = \frac{1}{|\text{Aut}(\mathcal{A})|} \sum_{\alpha \in \text{Aut}(\mathcal{A})} \left(\sum_{K \leq \mathcal{A}} \mu(K) \left(\sum_{\Omega \in S(K), \alpha(\Omega) = \Omega} x^{|\Omega|} \right) \right),$$

and

$$\Psi_{\mathcal{A}}(x) = \frac{1}{|\text{Inn}(\mathcal{A})|} \sum_{\alpha \in \text{Inn}(\mathcal{A})} \left(\sum_{K \leq \mathcal{A}} \mu(K) \left(\sum_{\Omega \in S(K), \alpha(\Omega) = \Omega} x^{|\Omega|} \right) \right).$$

3. DISTRIBUTION FOR EQUIVALENCE CLASSES

In this section, we will find a computation formula for the polynomial $\Psi_{\mathcal{A}}(x)$ when \mathcal{A} is a finite abelian group or the dihedral group D_n of order $2n$. If \mathcal{A} is abelian, then $\text{Inn}(\mathcal{A})$ is trivial and

$$\Psi_{\mathcal{A}}(x) = \sum_{K \leq \mathcal{A}} \mu(K) \left(\sum_{\Omega \in S(K)} x^{|\Omega|} \right),$$

by Theorem 2.5. It is not hard to show that

$$\sum_{\Omega \in S(K)} x^{|\Omega|} = \left((1 + x^2)^{\frac{|K| - |O_2(K)| - 1}{2}} (1 + x)^{|O_2(K)|} - 1 \right),$$

where $O_2(K) = \{g \in K : g^2 = e, g \neq e\}$. We summarize our discussion as follows.

Theorem 3.1. *For a finite abelian group \mathcal{A} ,*

$$\Psi_{\mathcal{A}}(x) = \sum_{K \leq \mathcal{A}} \mu(K) \left((1+x^2)^{\frac{|K|-|O_2(K)|-1}{2}} (1+x)^{|O_2(K)|} - 1 \right),$$

and hence

$$\mathcal{E}(\mathcal{A}) = \Psi_{\mathcal{A}}(1) = \sum_{K \leq \mathcal{A}} \mu(K) \left(2^{\frac{|K|+|O_2(K)|-1}{2}} - 1 \right).$$

Let n be a positive integer and μ be the number theoretical mu-function. Since a subgroup of the cyclic group \mathbb{Z}_n is also cyclic, say \mathbb{Z}_m with $m|n$ and that $\mu(\mathbb{Z}_m) = \mu(\frac{n}{m})$. Since $|O_2(\mathbb{Z}_m)| = \frac{1+(-1)^m}{2}$, we have the following corollary from Theorem 3.1.

Corollary 3.2. *For any positive integer n ,*

$$\Psi_{\mathbb{Z}_n}(x) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left((1+x^2)^{\lfloor \frac{d-1}{2} \rfloor} (1+x)^{\frac{1+(-1)^d}{2}} - 1 \right),$$

and hence

$$\mathcal{E}(\mathbb{Z}_n) = \Psi_{\mathbb{Z}_n}(1) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left(2^{\lfloor \frac{d}{2} \rfloor} - 1 \right).$$

Now we aim to compute $\Psi_{\mathbb{D}_n}(x)$, where $\mathbb{D}_n = \{a, b : a^n = 1, b^2 = 1, bab = a^{-1}\}$ is the dihedral group of order $2n$. Notice that $\text{Inn}(\mathbb{D}_n)$ is the set $\{\alpha_k, \beta_k : k = 1, 2, \dots, n\}$, where $\alpha_k(a) = a^{-k}aa^k = a$, $\alpha_k(b) = a^{-k}ba^k = ba^{2k}$, and $\beta_k(a) = (ba^k)a(ba^k) = a^{-1}$, $\beta_k(b) = (ba^k)b(ba^k) = ba^{2k}$ for each $k = 1, 2, \dots, n$. Each subgroup of the dihedral group \mathbb{D}_n is isomorphic to either \mathbb{Z}_m or \mathbb{D}_m for some $m|n$. There are exactly $\frac{n}{m}$ subgroups isomorphic to \mathbb{D}_m and only one subgroup isomorphic to \mathbb{Z}_m , where \mathbb{D}_1 is the subgroup generated by a reflection and \mathbb{D}_2 is a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, $\mu(\mathbb{Z}_m) = -\frac{n}{m}\mu(\frac{n}{m})$, $\mu(\mathbb{D}_m) = \mu(\frac{n}{m})$, and for each $m|n$ we can list the $\frac{n}{m}$ subgroups isomorphic to \mathbb{D}_m as follows: for each fixed $\ell = 1, 2, \dots, \frac{n}{m}$, $\mathbb{D}_m(\ell) = \{a^{s\frac{n}{m}}, ba^{s\frac{n}{m}+\ell} : s = 1, 2, \dots, m\}$. Then for each divisor m of n , we can see that

$$\begin{aligned} & \sum_{\Omega \in S(\mathbb{D}_m(\ell)), \alpha_k(\Omega) = \Omega} x^{|\Omega|} \\ &= \begin{cases} (1+x^2)^\alpha (1+x)^{\frac{1+(-1)^m}{2}} \left(1+x^{\frac{m}{(2km, m)}} \right)^{\left(\frac{2km}{n}, m \right)} - 1 & \text{if } 2k \equiv 0 \pmod{\frac{n}{m}}, \\ (1+x^2)^\alpha (1+x)^{\frac{1+(-1)^m}{2}} - 1 & \text{otherwise,} \end{cases} \end{aligned}$$

and that

$$\begin{aligned} & \sum_{\Omega \in S(\mathbb{D}_m(\ell)), \beta_k(\Omega) = \Omega} x^{|\Omega|} \\ &= \begin{cases} (1+x^2)^\alpha (1+x)^{\frac{1+(-1)^m}{2}} (1+x^2)^\alpha (1+x)^{\frac{3+(-1)^m}{2}} - 1 & \text{if } 2(k-\ell) \equiv 0 \pmod{\frac{n}{m}}, \\ (1+x^2)^\alpha (1+x)^{\frac{1+(-1)^m}{2}} - 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha = \lfloor \frac{m-1}{2} \rfloor$.

Similarly, we can see that

$$\begin{aligned} \sum_{\Omega \in S(\mathbb{Z}_m), \alpha_k(\Omega)=\Omega} x^{|\Omega|} &= \sum_{\Omega \in S(\mathbb{Z}_m), \beta_k(\Omega)=\Omega} x^{|\Omega|} = (1+x^2)^{\lfloor \frac{m-1}{2} \rfloor} (1+x)^{\frac{1+(-1)^m}{2}} - 1 \\ &= \begin{cases} (1+x^2)^{\frac{m-1}{2}} - 1 & \text{if } m \text{ is odd,} \\ (1+x^2)^{\frac{m-2}{2}} (1+x) - 1 & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

Now, Theorem 3.3 follows from the above discussion and Theorem 2.5.

Theorem 3.3. *Let \mathbb{D}_n be the dihedral group of order $2n$. Then we have*

$$\begin{aligned} 2n\Psi_{\mathbb{D}_n}(x) &= \sum_{k=1}^n \sum_{m|n} \left[-2\frac{n}{m} \mu\left(\frac{n}{m}\right) \left((1+x^2)^\gamma (1+x)^{\frac{1+(-1)^m}{2}} - 1 \right) \right. \\ &\quad \left. + \mu\left(\frac{n}{m}\right) \sum_{\ell=1}^{\frac{n}{m}} (F_{\alpha_k}(\ell, m) + F_{\beta_k}(\ell, m)) \right], \end{aligned}$$

where

$$\begin{aligned} F_{\alpha_k}(\ell, m) &= \begin{cases} (1+x^2)^\gamma (1+x)^{\frac{1+(-1)^m}{2}} \left(1+x^{\left(\frac{m}{\frac{2km}{n}, m}\right)} \right)^{\left(\frac{2km}{n}, m\right)} - 1 & \text{if } 2k \equiv 0 \pmod{\frac{n}{m}}, \\ (1+x^2)^\gamma (1+x)^{\frac{1+(-1)^m}{2}} - 1 & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} F_{\beta_k}(\ell, m) &= \begin{cases} (1+x^2)^\gamma (1+x)^{\frac{1+(-1)^m}{2}} (1+x^2)^\gamma (1+x)^{\frac{3+(-1)^m}{2}} - 1 & \text{if } 2(k-\ell) \equiv 0 \pmod{\frac{n}{m}}, \\ (1+x^2)^\gamma (1+x)^{\frac{1+(-1)^m}{2}} - 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\gamma = \lfloor \frac{m-1}{2} \rfloor$.

Corollary 3.4. *Let \mathbb{D}_n be the dihedral group of order $2n$. Now we have*

$$\mathcal{E}(\mathbb{D}_n) = \Psi_{\mathbb{D}_n}(1) = \sum_{m|n} \left(2, \frac{n}{m} \right) \mu\left(\frac{n}{m}\right) 2^{\lfloor \frac{m}{2} \rfloor} \left(2^{\left(\frac{2km}{n}, m\right)-1} + 2^{\lfloor \frac{m}{2} \rfloor} - 1 \right).$$

Proof. Since $\mathcal{E}(\mathbb{D}_n) = \Psi_{\mathbb{D}_n}(1)$, we have

$$\begin{aligned}
\mathcal{E}(\mathbb{D}_n) &= \frac{1}{2n} \sum_{m|n} \left[-2n \frac{n}{m} \mu\left(\frac{n}{m}\right) (2^{\lfloor \frac{m}{2} \rfloor} - 1) + \mu\left(\frac{n}{m}\right) \sum_{k=1}^n \sum_{\ell=1}^{\frac{n}{m}} (F_{\alpha_k}(\ell, m) + F_{\beta_k}(\ell, m)) \right] \\
&= \frac{1}{2n} \sum_{m|n} \left[-2n \frac{n}{m} \mu\left(\frac{n}{m}\right) (2^{\lfloor \frac{m}{2} \rfloor} - 1) \right. \\
&\quad + \mu\left(\frac{n}{m}\right) \sum_{\ell=1}^{\frac{n}{m}} \left((2, \frac{n}{m})m \left(2^{\lfloor \frac{m}{2} \rfloor + (\frac{2km}{n}, m)} + 2^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m+2}{2} \rfloor} \right) \right. \\
&\quad \left. \left. + \left(n - (2, \frac{n}{m})m \right) 2^{\lfloor \frac{m}{2} \rfloor + 1} - 2n \right) \right] \\
&= \frac{1}{2n} \sum_{m|n} \left[-2n \frac{n}{m} \mu\left(\frac{n}{m}\right) (2^{\lfloor \frac{m}{2} \rfloor} - 1) \right. \\
&\quad \left. + \frac{n}{m} \mu\left(\frac{n}{m}\right) \left((2, \frac{n}{m})m 2^{\lfloor \frac{m}{2} \rfloor} \left(2^{(\frac{2km}{n}, m)} + 2^{\lfloor \frac{m+2}{2} \rfloor} - 2 \right) + 2n 2^{\lfloor \frac{m}{2} \rfloor} - 2n \right) \right] \\
&= \sum_{m|n} (2, \frac{n}{m}) \mu\left(\frac{n}{m}\right) 2^{\lfloor \frac{m}{2} \rfloor} \left(2^{(\frac{2km}{n}, m)-1} + 2^{\lfloor \frac{m}{2} \rfloor} - 1 \right).
\end{aligned}$$

□

4. DEGREE DISTRIBUTION POLYNOMIALS FOR SOME CIRCULANT GRAPHS

In this section, we compute the degree distributions polynomials for the weak equivalence classes of some circulant graphs of prime power in subsection 4.1, of order $4p$ in subsection 4.2 and of square free order in subsection 4.3.

4.1. Degree distribution polynomials for circulant graphs of prime power order.

For a prime p , let \mathbb{Z}_{p^m} be the cyclic group of order p^m . For each $k = 0, 1, 2, \dots, m-1$, let $A_k = \{s \in \mathbb{Z}_{p^m} : (s, p^m) = p^k\}$, where (s, t) is the greatest common divisor of the positive integers s and t . Indeed, A_0 is $\text{Aut}(\mathbb{Z}_{p^m})$, the set of all automorphisms of \mathbb{Z}_{p^m} . Then $|A_k| = \phi(p^{m-k}) = p^{m-k-1}\phi(p)$, where ϕ is the Euler pi-function. Note that

$$G(\mathbb{Z}_{p^m}) = \{\Omega : -\Omega = \Omega, \Omega \subset \bigcup_{k=0}^{m-1} A_k\} - \{\Omega : -\Omega = \Omega, \Omega \subset \bigcup_{k=1}^{m-1} A_k\}.$$

For each $k = 0, 1, \dots, m-1$, let $X_k = \{\Omega : \Omega^{-1} = -\Omega = \Omega, \Omega \subset A_k\}$. Now X_k is an invariant subset of $\text{Aut}(\mathbb{Z}_{p^m})$ -action. For an automorphism α in $\text{Aut}(\mathbb{Z}_{p^m})$, we observe that

$$\sum_{\Omega \in \bigcup_{k=0}^{m-1} A_k, -\Omega = \Omega, \alpha(\Omega) = \Omega} x^{|\Omega|} = \left[\prod_{k=0}^{m-1} \left(1 + \sum_{\Omega \in X_k, \alpha(\Omega) = \Omega} x^{|\Omega|} \right) \right] - 1,$$

and

$$\sum_{\Omega \in \bigcup_{k=1}^{m-1} A_k, -\Omega = \Omega, \alpha(\Omega) = \Omega} x^{|\Omega|} = \left[\prod_{k=1}^{m-1} \left(1 + \sum_{\Omega \in X_k, \alpha(\Omega) = \Omega} x^{|\Omega|} \right) \right] - 1.$$

Thus, we have the following lemma.

Lemma 4.1. *Let α be an automorphism in $\text{Aut}(\mathbb{Z}_{p^m})$. Then we have*

$$\sum_{\Omega \in \text{Fix}_\alpha} x^{|\Omega|} = \left(\sum_{\Omega \in X_0, \alpha(\Omega)=\Omega} x^{|\Omega|} \right) \prod_{k=1}^{m-1} \left(1 + \sum_{\Omega \in X_k, \alpha(\Omega)=\Omega} x^{|\Omega|} \right).$$

Now, we aim to compute $\sum_{\Omega \in X_k, \alpha(\Omega)=\Omega} x^{|\Omega|}$ for each $\alpha \in \text{Aut}(\mathbb{Z}_{p^m})$ and each $k = 0, 1, \dots, m-1$. First we consider $p = 2$. Note that $\text{Aut}(\mathbb{Z}_{2^m})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and that the multiplicative group A_0 is equal to the group $\text{Aut}(\mathbb{Z}_{2^m})$. Let $\eta : A_0 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ be the isomorphism. Then $\eta(-1) = (1, 0)$ and hence, it naturally induces an isomorphism $\bar{\eta} : A_0/\mathbb{Z}_2 \rightarrow \mathbb{Z}_{2^{m-2}}$. So any subset Ω of A_0 satisfying $\Omega = -\Omega$ corresponds to a subset of $\mathbb{Z}_{2^{m-2}}$ and vice versa via an isomorphism $\bar{\eta}$, namely there is an one to one correspondence between X_0 and $\mathcal{P}(\mathbb{Z}_{2^{m-2}})$, where $\mathcal{P}(\mathbb{Z}_{2^{m-2}})$ is the powerset of $\mathbb{Z}_{2^{m-2}}$. Furthermore, $\text{Aut}(\mathbb{Z}_{2^m})/\mathbb{Z}_2$ -action on X_0 is equivalent to the natural $\mathbb{Z}_{2^{m-2}}$ -action on $\mathcal{P}(\mathbb{Z}_{2^{m-2}})$. Therefore, for each $\alpha \in A_0 = \text{Aut}(\mathbb{Z}_{2^m})$, we can see that

$$\sum_{\Omega \in X_0, \alpha(\Omega)=\Omega} x^{|\Omega|} = \sum_{\bar{\Omega} \in \bar{\eta}(X_0), \bar{\eta}(\alpha)(\bar{\Omega})=\bar{\Omega}} x^{2|\bar{\Omega}|} = \left(1 + x^{\frac{2^{m-1}}{(\bar{\eta}(\alpha), 2^{m-2})}} \right)^{(\bar{\eta}(\alpha), 2^{m-2})} - 1$$

and $\sum_{\Omega \in X_{m-1}, \alpha(\Omega)=\Omega} x^{|\Omega|} = x$. By a method similar to the case $k = 0$, for each $k = 1, 2, \dots, m-2$, we can see that

$$\sum_{\Omega \in X_k, \alpha(\Omega)=\Omega} x^{|\Omega|} = \left(1 + x^{\frac{2^{m-k-1}}{(\bar{\eta}(\alpha), 2^{m-k-2})}} \right)^{(\bar{\eta}(\alpha), 2^{m-k-2})} - 1.$$

Now, by Theorem 2.3 and Lemma 4.1,

$$\begin{aligned} 2^{m-1} \Psi_{\mathbb{Z}_{2^m}}^w(x) &= \sum_{\alpha \in \text{Aut}(\mathbb{Z}_{2^m})} \sum_{\Omega \in \text{Fix}_\alpha} x^{|\Omega|} \\ &= \sum_{\alpha \in \text{Aut}(\mathbb{Z}_{2^m})} \left[\left(1 + x^{\frac{2^{m-1}}{(\bar{\eta}(\alpha), 2^{m-2})}} \right)^{(\bar{\eta}(\alpha), 2^{m-2})} - 1 \right] \left(\prod_{k=1}^{m-2} \left(1 + x^{\frac{2^{m-k-1}}{(\bar{\eta}(\alpha), 2^{m-k-2})}} \right)^{(\bar{\eta}(\alpha), 2^{m-k-2})} \right) (1+x) \\ &= 2 \sum_{a \in \mathbb{Z}_{2^{m-2}}} \left[\left(1 + x^{\frac{2^{m-1}}{(a, 2^{m-2})}} \right)^{(a, 2^{m-2})} - 1 \right] (1+x) \prod_{k=1}^{m-2} \left(1 + x^{\frac{2^{m-k-1}}{(a, 2^{m-k-2})}} \right)^{(a, 2^{m-k-2})} \\ &= 2 \sum_{d|2^{m-2}} \phi(2^{m-2}/d) \left[\left(1 + x^{\frac{2^{m-1}}{d}} \right)^d - 1 \right] (1+x) \prod_{k=1}^{m-2} \left(1 + x^{\frac{2^{m-k-1}}{(d, 2^{m-k-2})}} \right)^{(d, 2^{m-k-2})} \\ &= 2 \sum_{\ell=0}^{m-2} \phi(2^{m-\ell-2}) \left[\left(1 + x^{2^{m-\ell-1}} \right)^{2^\ell} - 1 \right] (1+x) \prod_{k=1}^{m-2} \left(1 + x^{\frac{2^{m-k-1}}{(2^\ell, 2^{m-k-2})}} \right)^{(2^\ell, 2^{m-k-2})}. \end{aligned}$$

We summarize the above discussions as follow.

Theorem 4.2. *For each $m \geq 2$, we have*

$$2^{m-2} \Psi_{\mathbb{Z}_{2^m}}^w(x) = \sum_{t=0}^{m-2} \phi(2^t) \left(\left(1 + x^{2^{t+1}}\right)^{2^{m-t-2}} - 1 \right) (1+x) \prod_{s=t+1}^{m-2} (1+x^2)^{2^{m-s-2}} \prod_{s=1}^t \left(1 + x^{2^{t-s+1}}\right)^{2^{m-t-2}},$$

and hence

$$\begin{aligned} \mathcal{E}^w(\mathbb{Z}_{2^m}) &= \Psi_{\mathbb{Z}_{2^m}}^w(1) = \frac{1}{2^{m-2}} \sum_{t=0}^{m-2} \phi(2^t) \left(2^{1+2^{m-t-2}} - 2\right) \prod_{s=t+1}^{m-2} 2^{2^{m-s-2}} \prod_{s=1}^t 2^{2^{m-t-2}} \\ &= \frac{1}{2^{m-2}} \left[2^{2^{m-1}} (2^{2^{m-2}} - 1) + \sum_{t=1}^{m-2} 2^{(t+1)2^{m-t-2}+t-1} \left(2^{2^{m-t-2}} - 1\right) \right], \end{aligned}$$

where the product of the empty index set is defined to be 1.

Next we consider the case when p is an odd prime. It is well-known that there is an isomorphism $\theta : \text{Aut}(\mathbb{Z}_{p^m}) \rightarrow \mathbb{Z}_{p^{m-1}(p-1)}$. Since $\theta(-1) = \frac{p^{m-1}(p-1)}{2}$, we have an isomorphism $\bar{\theta} : \text{Aut}(\mathbb{Z}_{p^m})/\mathbb{Z}_2 \rightarrow \mathbb{Z}_{p^{m-1}(p-1)}/\mathbb{Z}_2 = \mathbb{Z}_{\frac{p^{m-1}(p-1)}{2}}$. It is also well-known that the multiplicative group A_0 is isomorphic to $\text{Aut}(\mathbb{Z}_{p^m})$. By a method similar to the case $p = 2$, there is an one to one correspondence between X_0 and $\mathcal{P}(\mathbb{Z}_{\frac{p^{m-1}(p-1)}{2}})$, where $\mathcal{P}(\mathbb{Z}_{\frac{p^{m-1}(p-1)}{2}})$ is the powerset of $\mathbb{Z}_{\frac{p^{m-1}(p-1)}{2}}$. Furthermore, $\text{Aut}(\mathbb{Z}_{p^m})/\mathbb{Z}_2$ -action on X_k is equivalent to the natural $\mathbb{Z}_{\frac{p^{m-1}(p-1)}{2}}$ -action on $\mathcal{P}(\mathbb{Z}_{\frac{p^{m-k-1}(p-1)}{2}})$ for each $k = 0, 1, 2, \dots, m-1$. For each $\alpha \in A_0 = \text{Aut}(\mathbb{Z}_{p^m})$, we can see that

$$\sum_{\Omega \in X_k, \alpha(\Omega)=\Omega} x^{|\Omega|} = \left(1 + x^{\frac{p^{m-k-1}(p-1)}{\left(\bar{\theta}(\alpha), \frac{p^{m-k-1}(p-1)}{2}\right)}} \right)^{\left(\bar{\theta}(\alpha), \frac{p^{m-k-1}(p-1)}{2}\right)} - 1.$$

By combining the above equation, Theorem 2.3 and Lemma 4.1, we obtain the following theorem.

Theorem 4.3. *For each odd prime p , we have*

$$\begin{aligned} \Psi_{\mathbb{Z}_{p^m}}^w(x) &= \frac{2}{p^{m-1}(p-1)} \sum_{d \mid \frac{\phi(p^m)}{2}} \phi\left(\frac{\phi(p^m)}{2d}\right) \left(\left(1 + x^{\frac{\phi(p^m)}{d}}\right)^d - 1 \right) \prod_{k=1}^{m-1} \left(1 + x^{\frac{\phi(p^{m-k})}{\left(d, \frac{\phi(p^{m-k})}{2}\right)}} \right)^{\left(d, \frac{\phi(p^{m-k})}{2}\right)}, \end{aligned}$$

and hence

$$\mathcal{E}^w(\mathbb{Z}_{p^m}) = \Psi_{\mathbb{Z}_{p^m}}^w(1) = \frac{2}{p^{m-1}(p-1)} \sum_{d \mid \frac{\phi(p^m)}{2}} \phi\left(\frac{\phi(p^m)}{2d}\right) (2^d - 1) \prod_{k=1}^{m-1} 2^{\left(d, \frac{\phi(p^{m-k})}{2}\right)}.$$

By a similar method, one can get the following result.

Theorem 4.4. *For each odd prime p , we have*

$$\begin{aligned} \Psi_{\mathbb{Z}_{2p^m}}^w(x) &= \frac{2}{p^{m-1}(p-1)} \sum_{d|\frac{\phi(p^m)}{2}} \phi\left(\frac{\phi(p^m)}{2d}\right) \left[\left(\left(1 + x^{\frac{\phi(p^m)}{d}}\right)^d - 1 \right) \left(1 + x^{\frac{\phi(p^m)}{d}}\right)^d \right. \\ &\quad \times \left(\prod_{k=1}^{m-1} \left(1 + x^{\frac{\phi(p^{m-k})}{\left(d, \frac{\phi(p^{m-k})}{2}\right)}} \right)^{2\left(d, \frac{\phi(p^{m-k})}{2}\right)} \right) (1+x) \\ &\quad + \left(\left(1 + x^{\frac{\phi(p^m)}{d}}\right)^d - 1 \right) \left\{ \left(\prod_{k=1}^{m-1} \left(1 + x^{\frac{\phi(p^{m-k})}{\left(d, \frac{\phi(p^{m-k})}{2}\right)}} \right)^{\left(d, \frac{\phi(p^{m-k})}{2}\right)} \right) (1+x) - 1 \right\} \\ &\quad \left. \times \prod_{k=1}^{m-1} \left(1 + x^{\frac{\phi(p^{m-k})}{\left(d, \frac{\phi(p^{m-k})}{2}\right)}} \right)^{\left(d, \frac{\phi(p^{m-k})}{2}\right)} \right] \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{E}^w(\mathbb{Z}_{2p^m}) &= \Psi_{\mathbb{Z}_{2p^m}}^w(1) = \frac{2}{p^{m-1}(p-1)} \sum_{d|\frac{\phi(p^m)}{2}} \phi\left(\frac{\phi(p^m)}{2d}\right) \left[2^{d+1} (2^d - 1) \prod_{k=1}^{m-1} 2^{\left(d, \frac{\phi(p^{m-k})}{2}\right)} \right. \\ &\quad \left. + (2^d - 1) \left(2 \prod_{k=1}^{m-1} 2^{\left(d, \frac{\phi(p^{m-k})}{2}\right)} - 1 \right) \prod_{k=1}^{m-1} 2^{\left(d, \frac{\phi(p^{m-k})}{2}\right)} \right]. \end{aligned}$$

4.2. Degree distribution polynomials for circulant graphs of order $4p$. In this subsection, we find the degree distribution polynomials for circulant graphs of order $4p$, where p is an odd prime number.

Let $n = 4p$ for some prime number p . For our convenience, we consider \mathbb{Z}_n as $\mathbb{Z}_4 \times \mathbb{Z}_p$. Now $\text{Aut}(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_{4p}^* \simeq (\mathbb{Z}_4^* \times \mathbb{Z}_p^*) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_{p-1})$. So any automorphism $\sigma \in \text{Aut}(\mathbb{Z}_n)$ can be identified with an element (j, k) in $\mathbb{Z}_2 \times \mathbb{Z}_{p-1}$. Let us consider $\mathbb{Z}_4 \times \mathbb{Z}_p$ as a disjoint union of the following subsets;

$$\begin{aligned} A_i &= \{(n_1, n_2) \mid n_1 = i \text{ and } n_2 \neq 0\}, \\ B_i &= \{(n_1, n_2) \mid n_1 = i \text{ and } n_2 = 0\}, \end{aligned}$$

where $i = 0, 1, 2, 3$. Note that $\text{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_p)$ can also be identified by $A_1 \cup A_3$ and all of the subsets $A_1 \cup A_3, A_0, A_2, B_1 \cup B_3, B_0, B_2$ are closed under $\text{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_p)$ -action.

Let σ be an element in $\text{Aut}(\mathbb{Z}_n)$ whose corresponding element $\mathbb{Z}_2 \times \mathbb{Z}_{p-1}$ is (j, k) with $(k, p-1) = d$. Now an orbit O_1 of σ in $A_1 \cup A_3$ satisfies $-O_1 = O_1$ if and only if $j = 1$,

$d \mid \frac{p-1}{2}$ and $\frac{p-1}{2d}$ is odd. Note that if $j = 1$ and $\frac{p-1}{d}$ is odd then $|O_2| = \frac{2p-2}{d}$, and if $j = 1$, $d \mid \frac{p-1}{2}$ and $\frac{p-1}{2d}$ is odd then $|O_2| = \frac{p-1}{d}$. For any $i = 0, 2$ and for any orbit O_2 of σ in A_i , we have $-O_2 = O_2$ if and only if $d \mid \frac{p-1}{2}$.

For any subset $\Omega \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$, Ω is a generating set of $\mathbb{Z}_4 \times \mathbb{Z}_p$ if and only if $\Omega \cap (A_1 \cup A_3) \neq \emptyset$ or $(\Omega \cap (A_0 \cup A_2) \neq \emptyset$ and $\Omega \cap (B_1 \cup B_3) \neq \emptyset$). So we have the following theorem.

Theorem 4.5. *For each odd prime p , we have*

$$\begin{aligned} \Psi_{\mathbb{Z}_{4p}}^w(x) &= \frac{1}{2p-2} \sum_{d \mid p-1} \phi\left(\frac{p-1}{d}\right) \left[\left\{ \left(1 + x^{\frac{2p-2}{d}}\right)^d + \left(1 + x^{\frac{2p-2}{\frac{p-1}{2}+d, p-1}}\right)^{\left(\frac{p-1}{2}+d, p-1\right)} \right\} \right. \\ &\quad \times \left(1 + x^{\frac{p-1}{\left(d, \frac{p-1}{2}\right)}}\right)^{2\left(d, \frac{p-1}{2}\right)} (1+x^2)(1+x) \\ &\quad \left. - 2 \left(1 + x^{\frac{p-1}{\left(d, \frac{p-1}{2}\right)}}\right)^{2\left(d, \frac{p-1}{2}\right)} (1+x) - 2(1+x^2)(1+x) + 2(1+x) \right], \end{aligned}$$

and hence

$$\mathcal{E}^w(\mathbb{Z}_{4p}) = \Psi_{\mathbb{Z}_{4p}}^w(1) = \frac{1}{2p-2} \sum_{d \mid p-1} \phi\left(\frac{p-1}{d}\right) \left[4^{\left(d, \frac{p-1}{2}\right)+1} \left(2^d + 2^{\left(\frac{p-1}{2}+d, p-1\right)}\right) - 2^{2\left(d, \frac{p-1}{2}\right)+2} - 4 \right].$$

Note that $\left(\frac{p-1}{2} + d, p-1\right)$ is $\frac{d}{2}$ if $\frac{p-1}{d}$ is odd; d if both $\frac{p-1}{d}$ and $\frac{p-1}{2d}$ are even; $2d$ if $\frac{p-1}{d}$ is even but $\frac{p-1}{2d}$ is odd.

4.3. Degree distribution polynomials for circulant graphs of square free order. In this subsection, we find the degree distribution polynomials for circulant graphs of square free order. First we consider the case that n is a product of three distinct primes; (1) $n = 2p_1p_2$ ($2 < p_1 < p_2$) and (2) $n = p_1p_2p_3$ ($2 < p_1 < p_2 < p_3$).

First, let $n = 2p_1p_2$ ($2 < p_1 < p_2$) be a square free natural number. Since $\text{Aut}(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1}$, any automorphism $\sigma \in \text{Aut}(\mathbb{Z}_n)$ can be identified with an element (n_1, n_2) in $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1}$. For $\sigma \in \text{Aut}(\mathbb{Z}_n)$ whose corresponding element in $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1}$ is (n_1, n_2) . Let $d_1 = (n_1, p_1 - 1)$, $d_2 = (n_2, p_2 - 1)$, $\beta_1 = (d_1, \frac{p_1-1}{2})$ and $\beta_2 = (d_2, \frac{p_2-1}{2})$. Now we have

$$\sum_{\Omega \in S(\mathbb{Z}_n), \sigma(\Omega) = \Omega} = (1+x) \left(1 + x^{\frac{p_1-1}{\left(d_1, \frac{p_1-1}{2}\right)}}\right)^{2\beta_1} \left(1 + x^{\frac{p_2-1}{\left(d_2, \frac{p_2-1}{2}\right)}}\right)^{2\beta_2} \left(1 + x^{\alpha(d_1, d_2)}\right)^{\frac{2(p_1-1)(p_2-1)}{\alpha(d_1, d_2)}},$$

where $\alpha(d_1, d_2)$ is defined as

$$\alpha(d_1, d_2) = \begin{cases} \text{lcm}\left(\frac{p_1-1}{d_1}, \frac{p_2-1}{d_2}\right) & \text{if } d_i \mid \frac{p_i-1}{2} \text{ and there exists a constant } c \text{ such} \\ & \text{that } cd_i \equiv \frac{p_i-1}{2} \pmod{p_i-1} \text{ for any } i = 1, 2, \\ 2 \text{lcm}\left(\frac{p_1-1}{d_1}, \frac{p_2-1}{d_2}\right) & \text{otherwise.} \end{cases}$$

Let $\gamma = \frac{(p_1-1)(p_2-1)}{\alpha(d_1, d_2)}$ then, we have

$$\begin{aligned}
\Psi_{\mathbb{Z}_n}^w(x) &= \frac{1}{|\text{Aut}(\mathbb{Z}_n)|} \sum_{\sigma \in \text{Aut}(\mathbb{Z}_n)} \left(\sum_{\mathbb{Z}_m \leq \mathbb{Z}_n} \mu(\mathbb{Z}_m) \left(\sum_{\Omega \in S(\mathbb{Z}_m), \sigma(\Omega)=\Omega} x^{|\Omega|} \right) \right) \\
&= \frac{1}{(p_1-1)(p_2-1)} \sum_{d_1|p_1-1, d_2|p_2-1} \phi\left(\frac{p_1-1}{d_1}\right) \phi\left(\frac{p_2-1}{d_2}\right) \\
&\quad \times \left[(1+x) \left(1+x^{\frac{p_1-1}{(d_1, \frac{p_1-1}{2})}}\right)^{2\beta_1} \left(1+x^{\frac{p_2-1}{(d_2, \frac{p_2-1}{2})}}\right)^{2\beta_2} (1+x^{\alpha(d_1, d_2)})^\gamma \right. \\
&\quad - (1+x) \left(1+x^{\frac{p_1-1}{(d_1, \frac{p_1-1}{2})}}\right)^{2\beta_1} - (1+x) \left(1+x^{\frac{p_2-1}{(d_2, \frac{p_2-1}{2})}}\right)^{2\beta_2} \\
&\quad - \left(1+x^{\frac{p_1-1}{(d_1, \frac{p_1-1}{2})}}\right)^{\beta_1} \left(1+x^{\frac{p_2-1}{(d_2, \frac{p_2-1}{2})}}\right)^{\beta_2} (1+x^{\alpha(d_1, d_2)})^\gamma \\
&\quad \left. + (1+x) + \left(1+x^{\frac{p_1-1}{(d_1, \frac{p_1-1}{2})}}\right)^{\beta_1} + \left(1+x^{\frac{p_2-1}{(d_2, \frac{p_2-1}{2})}}\right)^{\beta_2} - 1 \right].
\end{aligned}$$

For the next case, let $n = p_1 p_2 p_3$ ($2 < p_1 < p_2 < p_3$) be a square free natural number. Since $\text{Aut}(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1} \times \mathbb{Z}_{p_3-1}$, any automorphism $\sigma \in \text{Aut}(\mathbb{Z}_n)$ can be identified with an element (n_1, n_2, n_3) in $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1} \times \mathbb{Z}_{p_3-1}$. For $\sigma \in \text{Aut}(\mathbb{Z}_n)$ whose corresponding element in $\mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1} \times \mathbb{Z}_{p_3-1}$ is (n_1, n_2, n_3) , let $d_1 = (n_1, p_1-1)$, $d_2 = (n_2, p_2-1)$, $d_3 = (n_3, p_3-1)$, $\beta_1 = (d_1, \frac{p_1-1}{2})$, $\beta_2 = (d_2, \frac{p_2-1}{2})$, $\beta_3 = (d_3, \frac{p_3-1}{2})$, $\gamma_{12} = \frac{(p_1-1)(p_2-1)}{\alpha(d_1, d_2)}$, $\gamma_{13} = \frac{(p_1-1)(p_3-1)}{\alpha(d_1, d_3)}$ and $\gamma_{23} = \frac{(p_2-1)(p_3-1)}{\alpha(d_2, d_3)}$. Let $\alpha(d_1, d_2, d_3)$ be defined as

$$\alpha(d_1, d_2, d_3) = \begin{cases} \text{lcm}\left(\frac{p_1-1}{d_1}, \frac{p_2-1}{d_2}, \frac{p_3-1}{d_3}\right) & \text{if } d_i | \frac{p_i-1}{2} \text{ and there exists a constant } c \text{ such} \\ & \text{that } cd_i \equiv \frac{p_i-1}{2} \pmod{p_i-1} \text{ for any } i = 1, 2, 3, \\ 2 \text{ lcm}\left(\frac{p_1-1}{d_1}, \frac{p_2-1}{d_2}, \frac{p_3-1}{d_3}\right) & \text{otherwise.} \end{cases}$$

Let $\delta = \frac{(p_1-1)(p_2-1)(p_3-1)}{\alpha(d_1, d_2, d_3)}$. Now we have

$$\begin{aligned}
\sum_{\Omega \in S(\mathbb{Z}_n), \sigma(\Omega)=\Omega} &= \left(1+x^{\frac{p_1-1}{(d_1, \frac{p_1-1}{2})}}\right)^{\beta_1} \left(1+x^{\frac{p_2-1}{(d_2, \frac{p_2-1}{2})}}\right)^{\beta_2} \left(1+x^{\frac{p_3-1}{(d_3, \frac{p_3-1}{2})}}\right)^{\beta_3} \\
&\quad \times (1+x^{\alpha(d_1, d_2)})^{\gamma_{12}} (1+x^{\alpha(d_1, d_3)})^{\gamma_{13}} (1+x^{\alpha(d_2, d_3)})^{\gamma_{23}} (1+x^{\alpha(d_1, d_2, d_3)})^\delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi_{\mathbb{Z}_n}^w(x) &= \frac{1}{|\text{Aut}(\mathbb{Z}_n)|} \sum_{\sigma \in \text{Aut}(\mathbb{Z}_n)} \left(\sum_{\mathbb{Z}_m \leq \mathbb{Z}_n} \mu(\mathbb{Z}_m) \left(\sum_{\Omega \in S(\mathbb{Z}_m), \sigma(\Omega) = \Omega} x^{|\Omega|} \right) \right) \\
&= \frac{1}{(p_1 - 1)(p_2 - 1)(p_3 - 1)} \sum_{d_i | p_i - 1, i=1,2,3} \phi\left(\frac{p_1 - 1}{d_1}\right) \phi\left(\frac{p_2 - 1}{d_2}\right) \phi\left(\frac{p_3 - 1}{d_3}\right) \\
&\quad \times \left[\left(1 + x^{\frac{p_1 - 1}{(d_1, \frac{p_1 - 1}{2})}}\right)^{\beta_1} \left(1 + x^{\frac{p_2 - 1}{(d_2, \frac{p_2 - 1}{2})}}\right)^{\beta_2} \left(1 + x^{\frac{p_3 - 1}{(d_3, \frac{p_3 - 1}{2})}}\right)^{\beta_3} \right. \\
&\quad \times (1 + x^{\alpha(d_1, d_2)})^{\gamma_{12}} (1 + x^{\alpha(d_1, d_3)})^{\gamma_{13}} (1 + x^{\alpha(d_2, d_3)})^{\gamma_{23}} (1 + x^{\alpha(d_1, d_2, d_3)})^{\delta} \\
&\quad \left. - \sum_{1 \leq i < j \leq 3} \left(1 + x^{\frac{p_i - 1}{\beta_i}}\right)^{\beta_i} \left(1 + x^{\frac{p_j - 1}{\beta_j}}\right)^{\beta_j} (1 + x^{\alpha(d_i, d_j)})^{\gamma_{ij}} + \sum_{i=1}^3 \left(1 + x^{\frac{p_i - 1}{\beta_i}}\right)^{\beta_i} - 1 \right].
\end{aligned}$$

From now on, we consider general case. Let n be an odd square free number, namely $n = p_1 p_2 \cdots p_\ell$ for some distinct prime numbers p_1, p_2, \dots, p_ℓ . For each divisor d_i of $p_i - 1$ for $i = 1, 2, \dots, \ell$, we define $f(d_1, d_2, \dots, d_\ell)$ be the polynomial

$$(1 + x^{\alpha(d_1, d_2, \dots, d_\ell)})^{\frac{(p_1 - 1)(p_2 - 1) \cdots (p_\ell - 1)}{\alpha(d_1, d_2, \dots, d_\ell)}},$$

where $\alpha(d_1, d_2, \dots, d_\ell)$ is the number defined as follows:

$$\begin{cases} \frac{p_1 - 1}{(d_1, \frac{p_1 - 1}{2})} & \text{if } \ell = 1, \\ \text{lcm}\left(\frac{p_1 - 1}{d_1}, \frac{p_2 - 1}{d_2}, \dots, \frac{p_\ell - 1}{d_\ell}\right) & \text{if } \ell \geq 2, d_i | \frac{p_i - 1}{2}, \text{ there exists a constant } c \text{ such that} \\ & cd_i \equiv \frac{p_i - 1}{2} \pmod{p_i - 1} \text{ for any } i = 1, \dots, \ell, \\ 2 \text{lcm}\left(\frac{p_1 - 1}{d_1}, \frac{p_2 - 1}{d_2}, \dots, \frac{p_\ell - 1}{d_\ell}\right) & \text{otherwise.} \end{cases}$$

For convenience, for each d_i such that $d_i | p_i - 1$ for $i = 1, 2, \dots, \ell$, we define

$$F_n(d_1, d_2, \dots, d_\ell) = \prod_{k=0}^{l-1} \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq \ell} f(d_1, d_2, \dots, \widehat{d_{i_1}}, \dots, \widehat{d_{i_2}}, \dots, \widehat{d_{i_k}}, \dots, d_\ell),$$

where

$$f(d_1, \dots, \widehat{d_{i_1}}, \dots, \widehat{d_{i_k}}, \dots, d_\ell) = f(d_1, \dots, d_{i_1 - 1}, d_{i_1 + 1}, \dots, d_{i_k - 1}, d_{i_k + 1}, \dots, d_\ell).$$

Since $\text{Aut}(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_{p_1 - 1} \times \mathbb{Z}_{p_2 - 1} \times \cdots \times \mathbb{Z}_{p_\ell - 1}$, any automorphism $\sigma \in \text{Aut}(\mathbb{Z}_n)$ can be identified with an element $(n_1, n_2, \dots, n_\ell)$ in $\mathbb{Z}_{p_1 - 1} \times \mathbb{Z}_{p_2 - 1} \times \cdots \times \mathbb{Z}_{p_\ell - 1}$. Thus, we can see that

$$\sum_{\sigma \in \text{Aut}(\mathbb{Z}_n)} \sum_{\Omega \in S(\mathbb{Z}_n), \sigma(\Omega) = \Omega} x^{|\Omega|} = \sum_{(d_1, d_2, \dots, d_\ell) \in \Phi} \left(\prod_{i=1}^{\ell} \phi\left(\frac{p_i - 1}{d_i}\right) \right) F_n(d_1, \dots, d_\ell),$$

where $\Phi = \{(d_1, d_2, \dots, d_\ell) : d_i | (p_i - 1), 1 \leq i \leq \ell\}$. Similarly one can see that for any $m = p_1 \cdots p_{i_1-1} p_{i_1+1} \cdots p_{i_2-1} p_{i_2+1} \cdots p_{i_k-1} p_{i_k+1} \cdots p_\ell$, we have

$$\begin{aligned} & \sum_{\sigma \in \text{Aut}(\mathbb{Z}_n)} \sum_{\Omega \in S(\mathbb{Z}_m), \sigma(\Omega) = \Omega} x^{|\Omega|} \\ &= \sum_{(d_1, d_2, \dots, d_\ell) \in \Phi} \left(\prod_{i=1}^{\ell} \phi\left(\frac{p_i - 1}{d_i}\right) \right) F_m(d_1, \dots, d_{i_1-1} d_{i_1+1}, \dots, d_{i_k-1} d_{i_k+1}, \dots, d_\ell). \end{aligned}$$

For our convenience, we denote $F_m(d_1, \dots, d_{i_1-1} d_{i_1+1}, \dots, d_{i_k-1} d_{i_k+1}, \dots, d_\ell)$ simply by $F_{i_1, i_2, \dots, i_k}(d_1, \dots, d_\ell)$. Now the following theorem comes from the above discussions, Burnside Lemma and the principal of inclusion and exclusion.

Theorem 4.6. *Let $n = p_1 p_2 \cdots p_\ell$ be the product of any given ℓ distinct odd prime numbers $p_1 < p_2 < \dots < p_\ell$. Then we have*

$$\Psi_{\mathbb{Z}_n}^w(x) = \frac{1}{\phi(n)} \sum_{(d_1, d_2, \dots, d_\ell)} \left(\prod_{i=1}^{\ell} \phi\left(\frac{p_i - 1}{d_i}\right) \right) \left(\sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \ell} F_{i_1, i_2, \dots, i_k}(d_1, \dots, d_\ell) \right),$$

where d_i runs over all divisors of $p_i - 1$ for each $i = 1, 2, \dots, \ell$ and $F_{1, 2, \dots, \ell}(d_1, \dots, d_\ell)$ is considered to be 1.

For even square free number n , we have the following theorem by a similar way.

Theorem 4.7. *Let $n = 2p_1 p_2 \cdots p_\ell$ be a square free number, where p_1, p_2, \dots, p_ℓ are ℓ distinct prime numbers. Then we have*

$$\begin{aligned} \Psi_{\mathbb{Z}_n}^w(x) &= \frac{1}{\phi(n)} \sum_{(d_1, d_2, \dots, d_\ell)} \left(\prod_{i=1}^{\ell} \phi\left(\frac{p_i - 1}{d_i}\right) \right) [(-1)^\ell (1 + x) \\ &\quad + \sum_{k=0}^{\ell-1} (-1)^k \left(\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \ell} ((1 + x) F_{i_1, i_2, \dots, i_k}(d_1, \dots, d_\ell)^2 - F_{i_1, i_2, \dots, i_k}(d_1, \dots, d_\ell)) \right)], \end{aligned}$$

where d_i runs over all divisors of $p_i - 1$ for each $i = 1, 2, \dots, \ell$.

For $n \leq 20$, the degree distribution polynomials for circulant graphs up to equivalence are given in Table 1 and the degree distribution polynomials for circulant graphs up to weak equivalence are given in Table 2.

5. DEGREE DISTRIBUTION POLYNOMIALS FOR CAYLEY GRAPHS OF ORDER p OR $2p$

Throughout this section, we assume that p is an odd prime. It is well-known that any group of order p is isomorphic to the cyclic group \mathbb{Z}_p and that any group of order $2p$ is isomorphic to the cyclic group \mathbb{Z}_{2p} or the dihedral group \mathbb{D}_p . By Theorem 4.3, we have the following corollary.

n	$\Psi_{\mathbb{Z}_n}(x)$	$\mathcal{E}(\mathbb{Z}_n)$
2	x	1
3	x^2	1
4	$x^2 + x^3$	2
5	$2x^2 + x^4$	3
6	$x^2 + 2x^3 + x^4 + x^5$	5
7	$3x^2 + 3x^4 + x^6$	7
8	$2x^2 + 2x^3 + 3x^4 + 3x^5 + x^6 + x^7$	12
9	$3x^2 + 6x^4 + 4x^6 + x^8$	14
10	$2x^2 + 4x^3 + 5x^4 + 6x^5 + 4x^6 + 4x^7 + x^8 + x^9$	27
11	$5x^2 + 10x^4 + 10x^6 + 5x^8 + x^{10}$	31
12	$2x^2 + 2x^3 + 9x^4 + 9x^5 + 10x^6 + 10x^7 + 5x^8 + 5x^9 + x^{10} + x^{11}$	54
13	$6x^2 + 15x^4 + 20x^6 + 15x^8 + 6x^{10} + x^{12}$	63
14	$x^2 + 2x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 5x^8 + 5x^9 + 2x^{10} + 2x^{11} + x^{12} + x^{13}$	119
15	$6x^2 + 20x^4 + 35x^6 + 35x^8 + 21x^{10} + 7x^{12} + x^{14}$	125
16	$4x^2 + 4x^3 + 18x^4 + 18x^5 + 34x^6 + 34x^7 + 35x^8 + 35x^9 + 21x^{10} + 21x^{11} + 7x^{12} + 7x^{13} + x^{14} + x^{15}$	240
17	$8x^2 + 28x^4 + 56x^6 + 70x^8 + 56x^{10} + 28x^{12} + 8x^{14} + x^{16}$	255
18	$4x^2 + 5x^3 + 22x^4 + 28x^5 + 52x^6 + 56x^7 + 56x^8 + 69x^9 + 70x^{10} + 56x^{11} + 28x^{12} + 28x^{13} + 8x^{14} + 8x^{15} + x^{16} + x^{17}$	548
19	$9x^2 + 36x^4 + 84x^6 + 126x^8 + 126x^{10} + 84x^{12} + 36x^{14} + 9x^{16} + x^{18}$	511
20	$4x^2 + 4x^3 + 30x^4 + 30x^5 + 78x^6 + 78x^7 + 125x^8 + 125x^9 + 126x^{10} + 126x^{11} + 84x^{12} + 84x^{13} + 36x^{14} + 36x^{15} + 9x^{16} + 9x^{17} + x^{18} + x^{19}$	986

TABLE 1. The degree distribution polynomials $\Psi_{\mathbb{Z}_n}(x)$ for circulant graphs up to equivalence for $n \leq 20$.

Corollary 5.1. *For each odd prime p , the distribution polynomial for the weak equivalence classes of Cayley graphs of order p is*

$$\Psi_{\mathbb{Z}_p}^w(x) = \frac{2}{p-1} \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2d}\right) \left(\left(1 + x^{\frac{p-1}{d}}\right)^d - 1 \right),$$

and hence

$$\mathcal{E}^w(\mathbb{Z}_p) = \Psi_{\mathbb{Z}_p}^w(1) = \frac{2}{p-1} \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2d}\right) (2^d - 1).$$

By Theorem 4.6, we have the following.

Corollary 5.2. *For each odd prime p , we have*

$$\Psi_{\mathbb{Z}_{2p}}^w(x) = \left[\frac{2}{p-1} \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2d}\right) \left(1 + x^{\frac{p-1}{d}}\right)^d \left(\left(1 + x^{\frac{p-1}{d}}\right)^d (1+x) - 1 \right) \right] - x,$$

n	$\Psi_{\mathbb{Z}_n}^w(x)$	$\mathcal{E}^w(\mathbb{Z}_n)$
2	x	1
3	x^2	1
4	$x^2 + x^3$	2
5	$x^2 + x^4$	2
6	$x^2 + 2x^3 + x^4 + x^5$	5
7	$x^2 + x^4 + x^6$	3
8	$x^2 + x^3 + 2x^4 + 2x^5 + x^6 + x^7$	8
9	$x^2 + 2x^4 + 2x^6 + x^8$	6
10	$x^2 + 2x^3 + 3x^4 + 4x^5 + 2x^6 + 2x^7 + x^8 + x^9$	16
11	$x^2 + 2x^4 + 2x^6 + x^8 + x^{10}$	7
12	$x^2 + x^3 + 6x^4 + 6x^5 + 7x^6 + 7x^7 + 4x^8 + 4x^9 + x^{10} + x^{11}$	38
13	$x^2 + 3x^4 + 4x^6 + 3x^8 + x^{10} + x^{12}$	13
14	$x^2 + 2x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 5x^8 + 5x^9 + 2x^{10} + 2x^{11} + x^{12} + x^{13}$	43
15	$x^2 + 6x^4 + 11x^6 + 11x^8 + 7x^{10} + 3x^{12} + x^{14}$	40
16	$x^2 + x^3 + 5x^4 + 5x^5 + 10x^6 + 10x^7 + 11x^8 + 11x^9 + 7x^{10} + 7x^{11} + 3x^{12} + 3x^{13} + x^{14} + x^{15}$	76
17	$x^2 + 4x^4 + 7x^6 + 10x^8 + 7x^{10} + 4x^{12} + x^{14} + x^{16}$	35
18	$x^2 + 2x^3 + 7x^4 + 9x^5 + 18x^6 + 20x^7 + 25x^8 + 26x^9 + 20x^{10} + 20x^{11} + 10x^{12} + 10x^{13} + 4x^{14} + 4x^{15} + x^{16} + x^{17}$	178
19	$x^2 + 4x^4 + 10x^6 + 14x^8 + 14x^{10} + 10x^{12} + 4x^{14} + x^{16} + x^{18}$	59
20	$x^2 + x^3 + 9x^4 + 9x^5 + 25x^6 + 25x^7 + 38x^8 + 38x^9 + 39x^{10} + 39x^{11} + 27x^{12} + 27x^{13} + 13x^{14} + 13x^{15} + 4x^{16} + 4x^{17} + x^{18} + x^{19}$	314

TABLE 2. The degree distribution polynomials $\Psi_{\mathbb{Z}_n}(x)$ for circulant graphs up to weak equivalence for $n \leq 20$.

and hence

$$\mathcal{E}^w(\mathbb{Z}_{2p}) = \Psi_{\mathbb{Z}_{2p}}^w(1) = -1 + \frac{2}{p-1} \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2d}\right) 2^d (2^{d+1} - 1).$$

As the final part of this section, we will find the distribution polynomials for the weak equivalence classes of Cayley graphs whose underlying group is the dihedral group \mathbb{D}_p . It is well-known that $\text{Aut}(\mathbb{D}_p)$ is the set $\{\alpha_{ij} : i = 1, 2, \dots, p-1, j = 1, 2, \dots, p\}$, where $\alpha_{ij}(a) = a^i$ and $\alpha_{ij}(b) = ba^j$. For $i = 1$, it is not hard to show that

$$\sum_{\Omega \in S(\mathbb{D}_p), \alpha_{1j}(\Omega) = \Omega} x^{|\Omega|} = \begin{cases} (1+x^2)^{\frac{p-1}{2}}(1+x)^p - 1 & \text{if } j = p, \\ (1+x^2)^{\frac{p-1}{2}}(1+x^p) - 1 & \text{if } j \neq p, \end{cases}$$

$$\sum_{\Omega \in S(\mathbb{Z}_p), \alpha_{1j}(\Omega) = \Omega} x^{|\Omega|} = (1+x^2)^{\frac{p-1}{2}} - 1,$$

and

$$\sum_{\Omega \in S(\mathbb{D}_1), \alpha_{1j}(\Omega)=\Omega} x^{|\Omega|} = \begin{cases} x & \text{if } j = p, \\ 0 & \text{if } j \neq p. \end{cases}$$

For convenience, for each $i \in \text{Aut}(\mathbb{Z}_p)$, let i^* be the element in \mathbb{Z}_{p-1} corresponding to i under the isomorphism between $\text{Aut}(\mathbb{Z}_p)$ and \mathbb{Z}_{p-1} . Then $\frac{p-1}{(i^*, p-1)}$ and $\frac{p-1}{(i^*, \frac{p-1}{2})}$ have the order of i^* in \mathbb{Z}_{p-1} and $\mathbb{Z}_{p-1}/\mathbb{Z}_2$, respectively, where \mathbb{Z}_2 is the subgroup $\{0, \frac{p-1}{2}\}$ of \mathbb{Z}_{p-1} . For $i \neq 1$, it is also not hard to show that

$$\begin{aligned} \sum_{\Omega \in S(\mathbb{D}_p), \alpha_{ij}(\Omega)=\Omega} x^{|\Omega|} &= \left(1 + x^{\frac{p-1}{(i^*, \frac{p-1}{2})}}\right)^{(i^*, \frac{p-1}{2})} \left(1 + x^{\frac{p-1}{(i^*, p-1)}}\right)^{(i^*, p-1)} (1+x) - 1, \\ \sum_{\Omega \in S(\mathbb{Z}_p), \alpha_{ij}(\Omega)=\Omega} x^{|\Omega|} &= \left(1 + x^{\frac{p-1}{(i^*, \frac{p-1}{2})}}\right)^{(i^*, \frac{p-1}{2})} - 1, \end{aligned}$$

and

$$\sum_{\Omega \in S(\mathbb{D}_1), \alpha_{ij}(\Omega)=\Omega} x^{|\Omega|} = \begin{cases} x & \text{if } \mathbb{D}_1 = \{0\} \cup \{ba^{(i-1)^{-1}(-j)}\}, \\ 0 & \text{if otherwise,} \end{cases}$$

where $(i-1)^{-1}$ is the inverse in $\text{Aut}(\mathbb{Z}_p)$. Now, Theorem 5.3 follows from the above discussion and Theorem 2.5.

Theorem 5.3. *For each odd prime p , we have*

$$\begin{aligned} p(p-1)\Psi_{\mathbb{D}_p}^w(x) &= (1+x^2)^{\frac{p-1}{2}} [(1+x)^p + (p-1)x^p - 1] - px \\ &\quad + \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2}\right) \left[\left(1 + x^{\frac{p-1}{(d, \frac{p-1}{2})}}\right)^{(d, \frac{p-1}{2})} \left(\left(1 + x^{\frac{p-1}{d}}\right)^d (1+x) - 1 \right) - x \right], \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{E}^w(\mathbb{D}_p) &= \frac{1}{p(p-1)} \left[2^{\frac{p-1}{2}} (2^p + p - 2) - p + p \sum_{d \mid \frac{p-1}{2}} \phi\left(\frac{p-1}{2}\right) \left(2^{(d, \frac{p-1}{2})+d+1} - 2^{(d, \frac{p-1}{2})} - 1 \right) \right]. \end{aligned}$$

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